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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Tuesday 19 November 2013 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

Consider the following functions

$$f:\ ]1, +\infty[\to \mathbb{R}^+ \text{ where } f(x) = (x-1)(x+2)$$

$$g:\ \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R} \text{ where } g(x, y) = \big(\sin(x+y),\ x+y\big)$$

$$h:\ \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R} \text{ where } h(x, y) = (x+3y,\ 2x+y)$$

(a) Show that f is bijective. *[3]*

(b) Determine, with reasons, whether

(i) g is injective;

(ii) g is surjective. *[6]*

(c) Find an expression for $h^{-1}(x, y)$ and hence justify that h has an inverse function. *[5]*

2. *[Maximum mark: 11]*

(a) Let G be a group of order 12 with identity element e.

Let $a \in G$ such that $a^6 \neq e$ and $a^4 \neq e$.

(i) Prove that G is cyclic and state two of its generators.

(ii) Let H be the subgroup generated by a^4. Construct a Cayley table for H. *[9]*

(b) State, with a reason, whether or not it is necessary that a group is cyclic given that all its proper subgroups are cyclic. *[2]*

3. *[Maximum mark: 15]*

(a) Let A be the set of all 3×3 matrices of the form $\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where a and b are real numbers, and $a^2+b^2 \neq 0$.

(i) Show that $\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \dfrac{1}{a^2+b^2}\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$, $a^2+b^2 \neq 0$.

(ii) Hence prove that $(A, \times)$ is a group where $\times$ denotes matrix multiplication. (It may be assumed that matrix multiplication is associative). *[10]*

(b) Let B be the set of all 3×3 matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -d \\ 0 & d & c \end{pmatrix}$, where c and d are real numbers, and $c^2+d^2 \neq 0$.

Prove that the group $(B, \times)$ is isomorphic to the group $(A, \times)$. *[5]*

4. *[Maximum mark: 9]*

Let $(H, *)$ be a subgroup of the group $(G, *)$.

Consider the relation R defined in G by xRy if and only if $y^{-1} * x \in H$.

(a) Show that R is an equivalence relation on G. *[6]*

(b) Determine the equivalence class containing the identity element. *[3]*

5. *[Maximum mark: 11]*

(a) Given a set U, and two of its subsets A and B, prove that

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)\text{, where } A \setminus B = A \cap B'.$$ *[4]*

(b) Let $S = \{A, B, C, D\}$ where $A = \varnothing$, $B = \{0\}$, $C = \{0, 1\}$ and $D = \{0, 1, 2\}$.

State, with reasons, whether or not each of the following statements is true.

(i) The operation $\setminus$ is closed in S.

(ii) The operation $\cap$ has an identity element in S but not all elements have an inverse.

(iii) Given $Y \in S$, the equation $X \cup Y = Y$ always has a unique solution for X in S. *[7]*